CREAȚII MATEMATICE SERIA B
ARTICOLE ŞI STUDII ŞTIINȚIFICE

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Motto: *Eu sunt vița,voi sunteți mlădițele.Cel ce rămâne în Mine şi Eu în el,acela aduce roadă multă,căci fără Mine nu puteți face nimic(IOAN,cap.15,v.5).*

Probleme propuse:

1. Fie numerele reale $a,b>0$.Arătați că are loc dubla inegalitate:

$$1+\frac{M}{(1+a)^2\cdot(1+b)^2}\geq\frac{1+ab}{(1+a)^2}+\frac{1+ab}{(1+b)^2}\geq 1+\frac{m}{(1+a)^2\cdot(1+b)^2},$$

unde $m=\min\left\{\left(a^2-1\right)^2,\left(b^2-1\right)^2\right\}$ şi $M=\max\left\{\left(a^2-1\right)^2,\left(b^2-1\right)^2\right\}$.

Ion Bursuc,profesor,Suceava

Soluție

Observăm că: $A=\frac{1+ab}{(1+a)^2}+\frac{1+ab}{(1+b)^2}-1=\frac{1+ab}{(1+a)^2}-\frac{b}{a+b}+\frac{1+ab}{(1+b)^2}-\frac{a}{a+b}=$

$$=\frac{a+b+a^2b+ab^2-b-2ab-a^2b}{(1+a)^2\cdot(a+b)}+\frac{a+b+a^2b+ab^2-a-2ab-b^2a}{(1+b)^2\cdot(a+b)}=$$

$$=\frac{a(b-1)^2}{(1+a)^2(a+b)}+\frac{b(a-1)^2}{(1+b)^2(a+b)}=\frac{a\left(b^2-1\right)^2+b\left(a^2-1\right)^2}{(a+b)(1+a)^2(1+b)^2}\Rightarrow$$

$$\Rightarrow A\geq\frac{(a+b)m}{(a+b)(1+a)^2(1+b)^2}=\frac{m}{(1+a)^2(1+b)^2}\text{ şi}$$

$$A\leq\frac{(a+b)M}{(a+b)(1+a)^2(1+b)^2}=\frac{M}{(1+a)^2(1+b)^2}.$$

2. Fie numerele reale $a,b,c>0$. Arătați că are loc inegalitatea:

$$\frac{(a+b)\cdot(b+c)\cdot(c+a)}{8abc}\geq\frac{1}{2}\left(\frac{a+c}{b+c}+\frac{b+c}{a+c}\right)+\left|\frac{a-c}{a+c}\cdot\frac{b-c}{b+c}\right|.$$

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Soluție

Se observă că putem scrie succesiv: $\frac{(a+b)(b+c)(c+a)}{8abc}-\frac{1}{2}\left(\frac{a+c}{b+c}+\frac{b+c}{a+c}\right)=$

$$=\frac{(b+c)(c+a)}{8c}\left(\frac{1}{a}+\frac{1}{b}\right)-\frac{1}{2}\left(\frac{a+c}{b+c}+\frac{b+c}{a+c}\right)=\frac{(b+c)(c+a)}{8bc}-\frac{a+c}{2(b+c)}+$$

$$+\frac{(b+c)(c+a)}{8ac}-\frac{b+c}{2(a+c)}=\frac{a+c}{8bc(b+c)}(b-c)^2+\frac{b+c}{8ac(a+c)}(a-c)^2\geq$$

$$\geq\frac{(a+c)(b-c)^2}{2(b+c)^3}+\frac{(b+c)(a-c)^2}{2(a+c)^3}\geq\sqrt{\frac{(a+c)(b-c)^2}{(b+c)^3}\cdot\frac{(b+c)(a-c)^2}{(a+c)^3}}=\left|\frac{b-c}{b+c}\cdot\frac{a-c}{a+c}\right|.$$

3. Fie numerele reale $a,b,c>0$.

Arătați ca are loc

inegalitatea: $\frac{(a+b)(b+c)(c+a)}{8abc}\geq\frac{1}{2}+\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{a+b+c}{12abc}.$

$\min\left\{(a-b)^2,(b-c)^2,(c-a)^2\right\}.$

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Soluție

Folosim egalitatea de la problema precedentă

$\frac{(a+b)(b+c)(c+a)}{8abc}-\frac{1}{2}\left(\frac{a+c}{b+c}+\frac{b+c}{a+c}\right)=\frac{a+c}{8bc(b+c)}(b-c)^2+\frac{b+c}{8ac(a+c)}(a-c)^2$.Vom obține:

$$3\frac{(a+b)(b+c)(c+a)}{8abc}=\frac{1}{2}\left(\frac{a+c}{b+c}+\frac{b+c}{a+c}\right)+$$

$$+\frac{1}{2}\left(\frac{b+a}{c+a}+\frac{c+a}{b+a}\right)+\frac{1}{2}+\left(\frac{c+b}{a+b}+\frac{a+b}{c+b}\right)+\frac{(a+c)(b-c)^2}{8bc(b+c)}+\frac{(b+c)(a-c)^2}{8ac(a+c)}+$$

$$+\frac{(b+a)(c-a)^2}{8ca(c+a)}+\frac{(c+a)(b-a)^2}{8ba(b+a)}+\frac{(c+b)(a-b)^2}{8ab(a+b)}+\frac{(a+b)(c-b)^2}{8cb(c+b)}=$$

$$=\sum\frac{a+b+2c}{2(a+b)}+\sum\frac{(a+b+2c)(a-b)^2}{8ab(a+b)}=\sum\left(\frac{1}{2}+\frac{c}{a+b}\right)+\sum\frac{(a+b+2c)(a-b)^2}{8ab(a+b)}=$$

$$=\frac{3}{2}+\sum\frac{c}{a+b}+\sum\frac{(a+b+2c)(a-b)^2}{8ab(a+b)}=\frac{3}{2}+\sum\frac{c}{a+b}+\sum\frac{\left(c+\frac{2c^2}{a+b}\right)(a-b)^2}{8abc}\geq$$

$$\geq \frac{3}{2}+\sum \frac{c}{a+b}+\frac{m}{8abc}\left(\sum a+2\sum \frac{c^2}{a+b}\right)\geq \frac{3}{2}+\sum \frac{c}{a+b}+\frac{m}{8abc}\left(\sum a+\sum a\right)=$$

$$=\frac{3}{2}+\sum \frac{c}{a+b}+\frac{m\sum a}{4abc}\Rightarrow$$

$$\Rightarrow \frac{(a+b)(b+c)(c+a)}{8abc}\geq \frac{1}{2}+\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{m(a+b+c)}{12abc}.$$

Am folosit inegalitatea: $\sum \frac{a^2}{b+c}\geq \frac{a+b+c}{2}$.

4. Arătați că pentru orice $a,b>0$ are loc dubla inegalitate:

$\frac{a+b}{2}\geq \frac{a+b}{2}\left(\frac{2ab}{a^2+b^2}\right)^r\geq \sqrt{ab}$ dacă şi numai dacă $r\in\left[0,\frac{1}{4}\right]$.

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Soluție

Pentru orice $a,b>0$ are loc inegalitatea: $\frac{a+b}{2}\geq \frac{a+b}{2}\left(\frac{2ab}{a^2+b^2}\right)^r \Leftrightarrow$

$\Leftrightarrow \left(\frac{a^2+b^2}{2ab}\right)^r\geq 1\Leftrightarrow r\geq 0$. Vom arăta că pentru orice $a,b>0$ are loc inegalitatea:

$\frac{a+b}{2}\left(\frac{2ab}{a^2+b^2}\right)^r\geq \sqrt{ab}$ **(1)** $\Leftrightarrow r\leq \frac{1}{4}$**.**

Inegalitatea **(1)** se scrie astfel: $\left(\frac{2ab}{a^2+b^2}\right)^r\geq \frac{2\sqrt{ab}}{a+b},(\forall)a,b>0\Leftrightarrow$

$$\Leftrightarrow \left(\frac{a^2+b^2}{2ab}\right)^r\leq \frac{a+b}{2\sqrt{ab}},(\forall)a,b>0\Leftrightarrow \left(\frac{a^2+b^2}{2ab}\right)^{2r}\leq \frac{(a+b)^2}{4ab},(\forall)a,b>0\Leftrightarrow$$

$$\Leftrightarrow \left(1+\frac{(a-b)^2}{2ab}\right)^{2r}-1\leq \frac{(a-b)^2}{4ab},(\forall)a,b>0\Rightarrow \frac{\left(1+\frac{(a-b)^2}{2ab}\right)^{2r}-1}{\frac{(a-b)^2}{2ab}}\leq \frac{1}{2},(\forall)a,b>0,$$

$$a\neq b\Rightarrow \frac{\left(1+\frac{x^2}{2(a+x)a}\right)^{2r}-1}{\frac{x^2}{2(a+x)a}}\leq \frac{1}{2},(\forall)a,x>0\Rightarrow \lim_{x\searrow 0}\frac{\left(1+\frac{x^2}{2a(a+x)}\right)^{2r}-1}{\frac{x^2}{2a(a+x)}}\leq \frac{1}{2}\Rightarrow$$

$\Rightarrow 2r \le \frac{1}{2} \Rightarrow r \le \frac{1}{4}.$

Reciproc: dacă $r \le \frac{1}{4} \Rightarrow \left(\frac{2ab}{a^2+b^2}\right)^r \ge \left(\frac{2ab}{a^2+b^2}\right)^{\frac{1}{4}} \ge \frac{2\sqrt{ab}}{a+b}, (\forall) a,b>0,$ deoarece

inegalitatea $\sqrt[4]{\frac{2ab}{a^2+b^2}} \ge \frac{2\sqrt{ab}}{a+b}$ se scrie $\frac{2ab}{a^2+b^2} \ge \frac{16a^2b^2}{(a+b)^4} \Leftrightarrow$

$\Leftrightarrow (a+b)^4 \ge 8ab(a^2+b^2) \Leftrightarrow (a^2+b^2+2ab)^2 \ge 8ab(a^2+b^2) \Leftrightarrow (a^2+b^2-2ab)^2 \ge 0.$

5. Arătați că pentru orice $a,b>0$ are loc dubla inegalitate:

$$\frac{a+b}{2} \ge \left(\frac{a^2+b^2}{2ab}\right)^r \cdot \sqrt{ab} \ge \sqrt{ab} \Leftrightarrow r \in \left[0, \frac{1}{4}\right].$$

Ion Bursuc,profesor,Suceava

Soluție

Vom arăta că are loc inegalitatea $\frac{a+b}{2} \ge \left(\frac{a^2+b^2}{2ab}\right)^r \sqrt{ab}, (\forall) a,b>0 \Leftrightarrow$

$\Leftrightarrow r \le \frac{1}{4}.$

Dacă: $\frac{a+b}{2} \ge \left(\frac{a^2+b^2}{2ab}\right)^r \sqrt{ab}, (\forall) a,b>0,$ atunci ea se scrie astfel: $\frac{a+b}{2\sqrt{ab}} \ge \left(\frac{a^2+b^2}{2ab}\right)^r,$

$(\forall) a,b>0 \Leftrightarrow \frac{(a-b)^2}{4ab} \ge \left(\frac{a^2+b^2}{2ab}\right)^{2r} - 1, (\forall) a,b>0 \Leftrightarrow$

$$\Leftrightarrow \frac{1}{2} \ge \frac{\left(1+\frac{(a-b)^2}{2ab}\right)^{2r} - 1}{\frac{(a-b)^2}{2ab}}, (\forall) a,b>0 \Rightarrow$$

$$\Rightarrow \frac{1}{2} \ge \lim_{x \searrow 0} \frac{\left(1+\frac{x^2}{2a(a+x)}\right)^{2r} - 1}{\frac{x^2}{2a(a+x)}} = 2r \Rightarrow r \le \frac{1}{4}.$$

Reciproc:

Pentru $r = \frac{1}{4}$ inegalitatea $\frac{a+b}{2} \ge \left(\frac{a^2+b^2}{2ab}\right)^r \sqrt{ab}, (\forall) a,b>0$ devine:

$$\frac{(a+b)^4}{16} \geq \frac{a^2+b^2}{2ab} a^2b^2, (\forall) a,b>0 \Leftrightarrow (a+b)^4 \geq 8 \cdot ab(a^2+b^2), (\forall) a,b>0 \Leftrightarrow$$

$$\Leftrightarrow (a-b)^4 \geq 0, (\forall) a,b>0.$$

Pentru $r \leq \frac{1}{4}$ inegalitatea $\frac{a+b}{2} \geq \left(\frac{a^2+b^2}{2ab}\right)^r \sqrt{ab}, (\forall) a,b>0$ este adevarată

deoarece $\frac{a+b}{2} \geq \left(\frac{a^2+b^2}{2ab}\right)^{\frac{1}{4}} \sqrt{ab} \geq \left(\frac{a^2+b^2}{2ab}\right)^r \sqrt{ab}, (\forall) a,b>0$.

Inegalitatea din partea dreaptă are loc $\Leftrightarrow r \geq 0$.

6. Demontstrați dubla inegalitate:

$\frac{a+b}{2} \geq \frac{a+b}{2} \cdot \sqrt{\frac{(a+b)^2}{2(a^2+b^2)}} \geq \sqrt{ab}, (\forall) a,b>0.$ şi arătați că dacă

$\frac{a+b}{2} \geq \frac{a+b}{2}\left(\frac{(a+b)^2}{2(a^2+b^2)}\right)^r \geq \sqrt{ab}, (\forall) a,b>0,$ atunci $r \leq \frac{1}{2}$.

Ion Bursuc,profesor,Suceava

Soluție

Inegalitatea $\frac{a+b}{2} \geq \frac{a+b}{2}\sqrt{\frac{(a+b)^2}{2(a^2+b^2)}}$ este evident adevarată deoarece :

$$(a+b)^2 \leq 2(a^2+b^2), (\forall) a,b>0.$$

Inegalitatea din partea dreaptă se scrie astfel: $\frac{(a+b)^2}{4} \cdot \frac{(a+b)^2}{2(a^2+b^2)} \geq ab, (\forall) a,b>0 \Leftrightarrow$

$$\Leftrightarrow (a+b)^4 \geq 8ab(a^2+b^2), (\forall) a,b>0 \Leftrightarrow (a-b)^4 \geq 0, (\forall) a,b>0.$$

Dacă $\frac{a+b}{2}\left(\frac{(a+b)^2}{2(a^2+b^2)}\right)^r \geq \sqrt{ab}, (\forall) a,b>0,$ atunci $\left(\frac{2(a^2+b^2)}{(a+b)^2}\right)^r \leq \frac{a+b}{2\sqrt{ab}}, (\forall) a,b>0$

$$\Leftrightarrow \left(1+\frac{(a-b)^2}{(a+b)^2}\right)^{2r} - 1 \leq \frac{(a-b)^2}{4ab}, (\forall) a,b>0 \Rightarrow \frac{\left(1+\frac{x^2}{(2a+x)^2}\right)^{2r} - 1}{\frac{x^2}{(2a+x)^2}} \leq \frac{(2a+x)^2}{4a(x+a)},$$

$$(\forall)a,x>0\Rightarrow \lim_{x\to\infty}\frac{\left(1+\frac{x^2}{(2a+x)^2}\right)^{2r}-1}{\frac{x^2}{(2a+x)^2}}\le \lim_{x\to\infty}\frac{(2a+x)^2}{4a^2+4ax}\Rightarrow 2r\le 1\Rightarrow r\le\frac{1}{2}.$$

7. Demonstrați că pentru orice $a,b,c>0$ are loc inegalitatea:

$$\frac{(a+b)(b+c)(c+a)}{8abc}\ge\frac{(a+c)(b^2+c^2)}{(b+c)^3}+\frac{(b+c)(a^2+c^2)}{(a+c)^3}+\frac{(a-c)^2(b-c)^2}{(a+c)^2(b-c)^2}.$$

Ion Bursuc,profesor,Suceava

Soluție

Deoarece $\frac{(a+b)(b+c)(c+a)}{8abc}=\frac{(b+c)(c+a)}{8c}\left(\frac{1}{a}+\frac{1}{b}\right)=$

$=\frac{(b+c)(c+a)}{8ac}+\frac{(b+c)(c+a)}{8bc}$ putem scrie: $\frac{(a+b)(b+c)(c+a)}{8abc}-\frac{(a+c)(b^2+c^2)}{(b+c)^3}-$

$-\frac{(b+c)(a^2+c^2)}{(a+c)^3}=\frac{(b+c)(c+a)}{8bc}-\frac{(a+c)(b^2+c^2)}{(b+c)^3}+\frac{(b+c)(c+a)}{8ac}-\frac{(b+c)(a^2+c^2)}{(a+c)^3}=$

$=(a+c)\frac{\left((b+c)^4-8bc(b^2+c^2)\right)}{8bc(b+c)^3}+(b+c)\frac{\left((a+c)^4-8ac(a^2+c^2)\right)}{8ac(a+c)^3}=$

$=\frac{(a+c)(b-c)^4}{8bc(b+c)^3}+\frac{(b+c)(a-c)^4}{8ac(a+c)^3}\ge\frac{(a+c)(b-c)^4}{2(b+c)^5}+\frac{(b+c)(a-c)^4}{2(a+c)^5}\ge$

$\ge\sqrt{\frac{(b-c)^4(a-c)^4}{(b+c)^4(a+c)^4}}=\frac{(b-c)^2(a-c)^2}{(b+c)^2(a+c)^2}$,de unde rezultă inegalitatea din enunț.